Logic/Set Theory II - Ordinals and Cardinals

Christopher Strickland

March 3rd, 2011

Outline

Definitions

Definition

A linear well ordering $< P, \leq >$ is a well ordering if $\forall (A \subset P) \exists (p \in A) \forall (q \in A) p \leq q$. i.e., every subset has a least element.

Fact

AC implies the well-ordering theorem in first order logic (every set can be well-ordered).

Definition

Transfinite Induction: Suppose P is a well ordering and φ is a formula. If φ(base case) holds and $\forall (p \in P) \forall (q < p)[\varphi(q) \rightarrow \varphi(p)]$ then $\forall (p \in P) \varphi(p)$.

Definition

Transfinite Recursion: Suppose P is a well ordering and X is a set. Let Φ be an operator such that $\forall(p \in P)\forall(f$ a function such that $f : [q \in P : q < p] \to X)$, $\Phi(f) : [q \in P : q \leq p] \to X$ and $f \subseteq \Phi(f)$. Then there is a function $g : P \to X$ such that $\forall p \in P$, $g \restriction [q : q \leq p] = \Phi(g \restriction [q : q < p])$.

Fact

Every well ordering is isomorphic to an initial segment of O_n, the ordinal numbers.

Definition

A set x is transitive if $\forall(y \in x)$, $y \subset x$.

Example

$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$: transitive.
$\{\{\emptyset\}\}$: not transitive.

Definition

An ordinal is a hereditarily transitive set. That is, x is transitive and all of its elements are too. Thus, every element of an ordinal is an ordinal and ordinals are linearly ordered by the relation $\in$.

Basic Consequences

Fact

There is no set x such that all ordinals are elements of x. Thus, O_n is a class.

Proof.

If there was such an x, let $y = [z \in x : \text{ z is an ordinal }]$. So y is the set of all ordinals, y is a transitive set of ordinals, so it is an ordinal itself. $y \in y$. $\sharp$ □

Theorem

(ZFC) Every set can be well ordered. Observation: Suppose x is a set of nonempty sets and $\leq$ is a well ordering on $\bigcup x$. Then $f(y) =$ [the $\in$ smallest element] is a choice function.

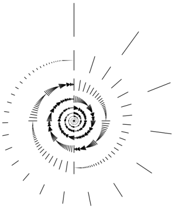
Definition

There are two types of ordinals: successor ordinals and limit ordinals.

- If an ordinal has a maximum element, it is a successor ordinal.
- If an ordinal is not zero and has no maximum element, it is a limit ordinal.

Example

$0, 1, 2, 3, \ldots \omega, \omega{+}1, \omega{+}2, \ldots \omega{\cdot}2, \omega{\cdot}2{+}1, \ldots \omega{\cdot}3, \ldots \omega^2, \ldots \omega^\omega, \ldots . \omega^{\omega^\omega}, \ldots$



Outline

Basics

Definition

- a and b have the same cardinality ($|a| = |b|$) if $\exists \pi : a \to b$ such that π is a bijection.
- Cardinality is an equivalence class.
- $|a| \leq |b|$ if there is an injection $\nu : a \to b$. This is transitive.

Theorem

(Schröder-Bernstein) $|a| \leq |b|$ *and* $|b| \leq |a|$ *implies* $|a| = |b|$.

Theorem

(Cantor) For every set x, $|\mathcal{P}(x)| \gneqq |x|$

Definition

A cardinal is an ordinal which has strictly stronger cardinality than all smaller ordinals.

Definition

A cardinal x is called transfinite if $\omega \leq x$.

Fact

There is no largest cardinal.

Fact

Trichotomy (either $x < y$, $x = y$, or $x > y$) of cardinal numbers is equivalent to AC.

Fact

- *Under AC, every cardinality has an ordinal representative.*
- *Without AC, nontrivial stuff happens...*

With AC, you can look at any cardinality, say $\mathcal{P}(\omega)$, and find a smallest ordinal equipotent with it.
Without AC, you must build from the bottom up:

$$0, 1, 2, ...\omega_0, ...\omega_1, ...\omega_2, ..., \omega_n, ...\omega_\omega, \omega_{\omega+1}, \omega_{\omega+2}, ...\omega_{\omega_1}, ...$$

where $\omega_\omega = sup_n(\omega_n)$, and for every ordinal α, ω_α is the α-th cardinal. Note: $\omega_{\omega_{\omega_{...}}}$ has a fixed point ω_κ.

Definition

There are two types of cardinals:

- Successors $\omega_{\alpha+1}$
- Limits ω_α where α is a limit ordinal

Definition

Cardinal arithmetic

- $|a| + |b| = |a \cup b|$
- $|a| \cdot |b| = |a \times b|$
- $|a|^{|b|} = |a^b = \{f : b \to a\}|$

Cardinal arithmetic under AC

- For transfinite cardinals k and λ, $k + \lambda = k \times \lambda = \max(k, \lambda)$.
- $k^\lambda = ?$

Notation

(ordinal) $\omega_\alpha = \aleph_\alpha$ (cardinality)

Continuum Hypothesis (CH)

$|2^\omega| = \aleph_1$

- (Gödel 1940) CH is undecidable in ZFC.
- Generalized Continuum hypothesis (GCH): $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$
- GCH is also undecidable in ZFC.
- (Sierpiński) ZF + GCH $\vdash$ AC

Cofinality

Definition

- A subset B of A is said to be cofinal if for every $a \in A$, there exists some $b \in B$ such that $a \leq b$.
- (AC) The cofinality of A, $cf(A)$, is the least of the cardinalities of the cofinal subsets of A. (Needs that there is such a least cardinal)
- This definition can be alternatively defined without AC using ordinals: $cf(A)$ is the least ordinal β such that there is a cofinal map π from x to A. This means that π has a cofinal image: $\forall(\gamma \in A)\exists(\delta \in \beta)$ s.t. $\pi(\delta) > \gamma$.
- We can similarly define cofinality for a limit ordinal α.

Example

- Let E_v denote the set of even natural numbers. E_v is cofinal in $\mathbb{N}$.
- $cf(E_v) = \omega$
- $cf(\omega^2) = \omega$

Definition

A limit ordinal α is regular if $cf(\alpha) = \alpha$.

Fact

1. *$cf(\alpha)$ is a cardinal.*
2. *$cf(cf(\alpha)) = cf(\alpha)$. So $cf(\alpha)$ is a regular cardinal.*
3. *Successor cardinals (ω_0, ω_1, etc.) are regular.*

Inaccessible cardinals

Definition

- κ is an inaccessible cardinal if it is a limit cardinal and regular.
- A cardinal κ is strongly inaccessible if it is inaccessible and closed under exponentiation (that is, $\kappa \neq 0$ and $\forall(\lambda < \kappa)$, $2^\lambda < \kappa$). Another way to put this is that κ cannot be reached by repeated powerset operations in the same way that a limit cardinal cannot be reached by repeated successor operations.

Theorem

ZFC $\nvdash$ there are strongly inaccessible cardinals other than $\aleph_0$.

von Neumann universe: V

Cumulative hierarchy

By recursion on $\alpha \in \mathrm{Ord}$, define V_α:

- $V_0 = 0$
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
- $V_\alpha = \bigcup_{\beta\in\alpha} V_\beta$

Theorem

Let κ be a strongly inaccessible cardinal. Then $V_\kappa \models ZFC$.

Proof.

Check the axioms.

1. Closure under powersets: $x \in V_\kappa \longrightarrow \mathcal{P}(x) \in V_\kappa$
2. Axiom of union: $x \in V_\kappa$ implies that $x \in V_\alpha$ for some $\alpha \in \kappa$, $x \subseteq V_\alpha$. $V_{\alpha+1}$ contains all subsets of x. Thus $\{y \in V_{\alpha+1} : y \subset x\} = \mathcal{P}(x) \in V_{\alpha+1}$, and since x was arbitrary, this is exactly the axiom of union.
3. Axiom of replacement (Suppose a function h is definable in V_κ. $dom(h) \in V_\kappa \longrightarrow rng(h) \in V_\kappa$): Let $g : dom(h) \longrightarrow \kappa$ be definied by $g(x) =$ least(α) s.t. $h(x) \in V_\alpha$. $dom(h) \in V_\beta$ for some $\beta \in \kappa$, so $|dom(h)| < \kappa$, and so $rng(g) \subseteq \gamma$ for some $\gamma \in \kappa$. $rng(h) \subseteq V_\gamma$ implies that $rng(h) \in V_{\gamma+1}$, so $rng(h) \in V_\kappa$.

□

Outline

Constructible (Gödel) Universe: L

L is built in stages, resembling V. The main difference is that instead of using the powerset of the previous stage, L takes only those subsets which are definable by a formula with parameters and quantifiers.

The constructible hierarchy

- $L_\alpha : \alpha \in$Ord
- $L_0 = \emptyset$
- $L_{\alpha+1}$ =definable (with parameters) subsets in the logical structure $< L_\alpha, \in >$.
- $L_\alpha = \bigcup_{\beta\in\alpha} L_\beta$

Constructible (Gödel) Universe: L

Gödel proved:

1. $L \models \text{ZFC} + \text{CH}$
2. $L \models$ there is a definable well ordering of the reals.
3. $L \models$ there is a coanalytic uncountable set with no perfect subset.

Additionally, L is the smallest transitive model of ZFC which contains all ordinals.

Axiom of constructibility (V=L)

Every set is constructible.

One can show that if ZFC is consistent, then so is ZFC + V=L, and then so is ZFC + CH. Thus CH is not disprovable in ZFC. Looking for a contradiction is the same as looking for a contradiction of ZFC. But this does not prove the independence of CH which was claimed earlier. What about $\neg$CH?

Forcing: Proving the independence of CH in ZFC

Three possible approaches:

1. Boolean valued approach (skip)
2. Model-theoretic approach. There is a countable model of ZFC, M. Add an "ideal point" G, and construct a model M[G] of ZFC. This is analogous to extending a field.
3. Axiomatic approach. Uses additional axioms like Martin's axiom (MA). ZFC + MA is consistent, which is proved by forcing. Then you derive consequences from these axioms. This is often used by mathematicians who don't know forcing... Logicians use 2 to get these axioms, and then others can apply them.

Martin's axiom

Definition

A subset A of a poset X is said to be a strong antichain if no two elements have a common lower bound (i.e. they are incompatible).

Definition

A poset X is said to satisfy the countable chain condition or be ccc if every strong antichain in X is countable.

Definition

Let P be a poset. $F \subseteq P$ is a filter if:

1. $\forall (p \in F)\ \forall (q \geq p) q \in F$ (upwards closed)
2. $\forall (p \in F)\ \forall (q \in F)\ \exists (r \in F)\ r \leq p,\ r \leq q$ (no incompatibility. there is a common element down the chain.)

Martin's axiom

Martin's axiom (MA_κ)

Let κ be a cardinal. For every ccc partial order P and every collection of open dense sets $\{D_\alpha : \alpha \in \kappa\}$, there is a filter $G \subset P$ such that $\forall(\alpha \in \kappa)\ G \cap D_\alpha \neq \emptyset$.
MA (no subscript) says that MA_κ holds for every $\kappa < 2^{\aleph_0}$.

Definition

A subset A of X is meager if it is the union of countably many nowhere dense subsets in X.

Fact

The closure of a nowhere dense subset is nowhere dense.

Martin's axiom

Theorem

($MA_{\aleph_1}$) 2^ω cannot be covered by $\aleph_1$ many meager sets.

Proof.

Let $P = 2^{<\omega}$ (finite binary paths, ordered by inclusion, $\in$). This is ccc. Suppose that $\{C_\alpha : \alpha \in \omega_1\}$ are closed, nowhere dense subsets of 2^ω (infinite or finite binary paths). I will find a sequence $x \in 2^\omega \setminus \bigcup_{\alpha \in \omega_1} C_\alpha$. Let $D_\alpha = \{t \in 2^{<\omega} : O_t \cap C_\alpha = \emptyset\}$, where O_t is the open set in 2^ω defined by all sequences extending t. $D_\alpha \subseteq 2^{<\omega}$, and note that D_α is open dense in $2^{<\omega}$. $MA_{\aleph_1}$ implies there is a filter G in $2^{<\omega}$ meeting all sets $D_\alpha : \alpha \in \omega_1$. Let $x = \bigcup G$. The filter has ω paths, so $x \in 2^\omega$. Since it hits each D_α, those extensions will force x to miss every C_α. So $x \notin \bigcup_{\alpha \in \omega_1} C_\alpha$. □

Martin's axiom

Corollary

This proves the negation of the continuum hypothesis.

Furthermore:

Fact

($MA_{\aleph_1}$)

- *The union of $\aleph_1$ meager sets is meager.*
- $[0,1]$ *cannot be covered by $\aleph_1$ many sets of Lebesgue measure zero.*
- *If $\{A_\alpha : \alpha \in \omega_1\}$ are measure 0 sets, then $\bigcup_{\alpha\in\omega_1} A_\alpha$ is measure zero.*

($MA_{\aleph_\kappa}$) $2^\kappa = 2^\omega$

Suslin's problem

Theorem

Suppose $< K, \leq >$ *is:*

1. *dense, no endpoints*
2. *complete*
3. *separable*

then K *is order-isomorphic to* $\mathbb{R}$.

Suslin's problem

Suppose $< K, \leq >$ is (1) and (2) and
(3*) Every collection of pairwise disjoint intervals is countable (ccc).
Does it still follow that K is order-isomorphic to $\mathbb{R}$?

Suslin's problem

Answer

Undecidable in ZFC
Undecidable in ZFC + GCH *and* ZFC + ¬CH
Yes under $MA_{\aleph_1}$
No under $V = L$

Definition

$< S, \leq >$ is a Suslin line if it satisfies (1), (2), and (3*), not separable. The Suslin hypothesis says that there are no Suslin lines - they are all isomorphic to the real line.

Questions?